CSCI 1311: Midterm 1 Review Worksheet
[SOLUTIONS]

10 Feb. 2020

---

1. Let the universe U be the set of GW Students. Let T be the set of GW students living in Thurston Hall, S be the set of students majoring in computer science, and V all the set of students in SEAS. Write the English sentences in set-notation and the set-notation in English

   (a) Set of CS majors that also live in Thurston Hall.

   $T \cap S$

   (b) Set of GW students not in SEAS.

   V^c

   (c) Set of CS majors not living in Thurston Hall.

   $T^c \cap S$

   (d) $T \cup V$

   Students living in Thurstan or in SEAS

   (e) $T \cap V$

   Students living in Thurstan and also in SEAS

   (f) $(T \cup S \cup V)^C$

   Students that are not living in Thurstan, CS majors, and in SEAS.

2. Consider the following sets

$$A = \{\emptyset, 2, 5, 9\}$$
$$B = \{x \in \mathbb{Z} \mid 2 \leq x \leq 13 \land \text{Prime(x)}\}$$
$$C = \{42, 64, 128\}$$

   (a) In set roster, what is $A \cap B$?

$\{2, 5\}$

(b) In set roster, what is $A \cup B$?

$\{\emptyset, 2, 3, 5, 7, 9, 11, 13\}$

(c) In set roster, what is $\mathcal{P}(C)$?

$\{\emptyset, \{42\}, \{64\}, \{128\}, \{42, 64\}, \{42, 128\}, \{64, 128\}, \{42, 64, 128\}\}$

(d) What is the value of $|A|$?

4

(e) True/False: $\emptyset \subseteq C$

True

(f) True/False: $\emptyset \subseteq A$

True

(g) True/False: $\{\emptyset\} \subseteq A$

True

(h) In set roster, what is $B - A$?

$\{3, 7, 11, 13\}$

3. Let $P(x, y)$ be true if person x and person y are friends and $x \neq y$. Let $Q(x)$ be true if person x goes to GW. Convert the English statements to quantified, prepositional logic.

(a) There are three people who are friends with each other..

$(\exists x, y, z)(P(x, y) \wedge P(y, z) \wedge P(x, z))$

(b) Every person has at least one friend

$(\forall x)((\exists y)(P(x, y))$

(c) There is a person who is friends with everyone.

$(\exists x)((\forall y)(P(x, y))$

(d) Every person has no more than two friends.

$(\forall x)(\exists a, b)[(\exists y)(P(x, y)) \implies (y = a \vee y = b)]$

4. Draw a truth table for the following logical expressions

(a) $p \lor q \land r$

p	q	r	$q \land r$	$p \lor q \land r$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T
T	F	F	F	T
T	F	T	F	T
T	T	F	F	T
T	T	T	T	T

(b) $p \to (q \lor \neg r)$

p	q	r	$\neg r$	$q \lor \neg r$	$p \to (q \lor \neg r)$
F	F	F	T	T	T
F	F	T	F	F	T
F	T	F	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	F	T	F	F	F
T	T	F	T	T	T
T	T	T	F	T	T

5. Prove the following with equivalent statments

(a) $(p \land q) \to r \equiv (p \to r) \lor (q \to r)$

$$
\begin{aligned}
p \land q \to r &\equiv \neg(p \land q) \lor r \\
&\equiv \neg p \lor \neg q \lor r \\
&\equiv \neg p \lor \neg q \lor (r \lor r) \\
&\equiv (\neg p \lor r) \lor (\neg q \lor r) \\
&\equiv (p \to r) \lor (q \to r)
\end{aligned}
$$

(b) $(p \land q) \lor (r \land \neg q) \equiv (p \lor r) \land (p \lor \neg q) \land (q \lor r)$

$$
\begin{aligned}
(p \land q) \lor (r \land \neg q) &\equiv \neg(\neg p \lor \neg q) \lor \neg(\neg r \lor q) \\
&\equiv \neg((\neg p \lor \neg q) \land (\neg r \lor q)) \\
&\equiv \neg((\neg p \land (\neg r \lor q)) \lor (\neg q \land (\neg r \lor q))) \\
&\equiv \neg((\neg p \land \neg r) \lor (\neg p \land q) \lor (\neg q \land \neg r) \lor (\neg q \land q)) \\
&\equiv \neg((\neg p \land \neg r) \lor (\neg p \land q) \lor (\neg q \land \neg r) \lor \mathbf{c}) \\
&\equiv \neg((\neg p \land \neg r) \lor (\neg p \land q) \lor (\neg q \land \neg r)) \\
&\equiv \neg(\neg p \land \neg r) \land \neg(\neg p \land q) \land \neg(\neg q \land \neg r) \\
&\equiv (p \lor r) \land (p \lor \neg q) \land (q \lor r)
\end{aligned}
$$

6. Prove the following statements

   (a) If $a|b$, then $a^2|b^2$.

   If $a|b$, then there exists c such that $b = ac$. Then $b^2 = a^2c^2$, and so $a^2|b^2$ because exists a integer (namely c^2) such $a^2c^2 = b^2$.

   (b) All primes $p > 2$ can be written as $3q + 1$ or $3q - 1$.

   By the quotient-remainder theorem, all integers n, there exists a q such that

$$n = 3q \quad \vee \quad n = 3q + 1 \quad \vee \quad n = 3q + 2$$

   Note that $n = 3q - 1 = 3q + 2 - 3 = 3(q - 1) + 2$, and then let $q' = q - 1$. So $3q - 1$ is another form of $3q + 2$.

   By cases, $n = 3q$ is divisible by 3, so none of the integers n described by that case can be prime, so all the primes must be in the remaining cases $n = 3q + 1$ and $n = 3q + 2$ ($n = 3q - 1$)

7. Proove by contradiction

   (a) The square root of an irrational number is irrational.

   Proof by contradiction. Assume that an irrational number r has a square root that is rational,

$$\sqrt{r} = \frac{a}{b}$$
$$r = \frac{a^2}{b^2}$$

   But this is a contradiction because r is irrationally and cannot be represented as a fraction. Thus the square root of a irrational number is also irrational

8. Prove by contrapositive

   (a) If $m + n$ is even, then either m and n are both even or m and n are both odd.

   By contra position, we can sow that if n is odd or m is even, or n is even and n is even or m is odd, then $m + n$ is even.

   If n is odd and m is even, then $n + m = (2a + 1) + (2b) = 2(a + b) + 1$ and is odd

   If n is even and m is odd, then $n + m = (2a) + (2b + 1) = 2(a + b) + 1$ and is odd.

9. Prove the following using weak induction.

   (a) For all $n \geq 1$, $\sum_{i=1}^{n} 4(i + 1) = 2n(n + 3)$

   Base case: $n = 1$, then $4(1 + 1) = 8 = 2 \cdot 1(1 + 3)$

   Inductive step: Assume $\sum_{i=1}^{n} 4(i+1) = 2n(n+3)$, and show that $\sum_{i=1}^{n+1} 4(i+1) = 2(n+1)((n+1)+3)$

   Working on the left hand side of our To show

$$\sum_{i=1}^{n+1} 4(i + 1) = 4((n + 1) + 1) + \sum_{i=1}^{n} 4(i + 1)$$

And by IH

$$\begin{aligned}4((n+1)+1)+\sum_{i=1}^{n}4(i+1) &= 4(n+1+1)+2n(n+3)\\ &= 4n+8+2n^2+6n\\ &= 2n^2+10n+8\end{aligned}$$

And working on the right hand side:

$$\begin{aligned}2(n+1)((n+1)+3) &= 2(n+1)(n+4)\\ &= 2(n^2+5n+4)\\ &= 2n^2+10n+8\end{aligned}$$

(b) For all integers $n \geq 1$, $6|(7^n-1)$

Base case: $n=1$, then $7^1-1=6$ and $6|6$

Inductive Step: Assume $6|(7^n-1)$, show that $6|(7^{n+1}-1)$.

Using the IH, if $6|(7^n-1)$, then there exists an a such that $6a=7^n-1$.

$$\begin{aligned}6a &= 7^n-1\\ 7\cdot 6a &= 7(7^n-1)\\ 42a &= 7^{n+1}-7\\ 42a &= 7^{n+1}-(6+1)\\ 42a &= 7^{n+1}-6-1\\ 42a+6 &= 7^{n+1}-1\\ 6(7a+1) &= 7^{n+1}-1\end{aligned}$$

So $6|(7^{n+1}-1)$

10. Prove the following using strong induction.

(a) For all integers $n \geq 8$, there exists integers a and b such that $n = 4a+5b$.

Base Case: $n=8$, then $a=2$ and $b=0$ and $8=4\cdot 2$

Inductive step: Assume that forall $8 \leq m \leq n$, there exists an a and b such that $m=4a+5b$. Show that exists a' and b' such that $n+1=4a'+5b'$

By cases of even/odd on $n+1$

if $n+1$ is even, then $n+1 = 2d$. By the IH, since $d < n$

$$\begin{aligned} n+1 &= 2d \\ &= 2(4a+5b) \\ &= 4(2a)+5(2b) \end{aligned}$$

So $a' = 2a$ and $b' = 2b$

if $n+1$ is odd, then $n+1 = 2d+1$. By the IH, since $d < n$ because $n = 2d$

$$\begin{aligned} n+1 &= 2d+1 \\ &= 2(4a+5b)+1 \\ &= 4(2a)+5(2b)+5-4 \\ &= 4(2a)-4+5(2b)+5 \\ &= 4(2a-1)+5(2b+1) \end{aligned}$$

So $a' = 2a-1$ and $b' = 2b+1$