CSCI 1311: Midterm 2 Review Worksheet
[SOLUTIONS]

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1. Solve the recurrence relation such that you can write t_n in terms of n alone.

   (a) $t_n = t_{n-1} + 1 \qquad t_0 = 1$

$$
\begin{aligned}
t_n &= t_{n-1} + 1 \quad t_0 = 1 \\
&= t_{n-2} + 1 + 1 \\
&= t_{n-3} + 1 + 1 + 1 \\
&= t_{n-i} + i \\
i = n & \\
&= t_0 + n \\
&= \boxed{1 + n}
\end{aligned}
$$

   (b) $t_n = 5t_{n-1} + 4 \qquad t_0 = 11$

$$
\begin{aligned}
t_n &= 5t_{n-1} + 4 \\
&= 5(5t_{n-2} + 4) + 4 \\
&= 5^2 t_{n-2} + 5 \cdot 4 + 4 \\
&= 5^2(5t_{n-3} + 4) + 5 \cdot 4 + 4 \\
&= 5^3 t_{n-3} + 5^2 \cdot 4 + 5^1 \cdot 4 + 5^0 \cdot 4 \\
&= 5^i t_{n-i} + 4 \cdot \sum_{j=0}^{i-1} 5^i
\end{aligned}
$$

$n = i$

$$
\begin{aligned}
&= 5^n t_0 + 4 \cdot \overbrace{\sum_{j=0}^{n-1} 5^i}^{\text{geometric sum}} \\
&= 5^n \cdot 11 + 4\left(\frac{1 - 5^n}{1 - 5}\right) \\
&= 5^n \cdot 11 + 4\left(\frac{5^n - 1}{4}\right) \\
&= 5^n \cdot 11 + 5^n - 1 \\
&= \boxed{12 \cdot 5^n - 1}
\end{aligned}
$$

(c) $t_n = t_{n-1} + 2n \qquad t_0 = 2$

$$
\begin{aligned}
t_n &= t_{n-1} + 2n \\
&= t_{n-2} + 2(n-1) + 2n \\
&= t_{n-3} + 2(n-2) + 2(n-1) + 2n \\
&= t_{n-3} + 2((n-2) + (n-1) + n) \\
&= t_{i-3} + 2((n - (i-1)) + (n - (i-2)) + \ldots + (n-1) + (n-0)) \\
&= t_{n-i} + 2 \cdot \sum_{j=0}^{i-1} (n-j)
\end{aligned}
$$

$i = n$

$$
\begin{aligned}
&= t_0 + 2 \cdot \sum_{j=0}^{n-1} (n-j) \\
t_n &= 2 + 2 \cdot \sum_{j=0}^{n-1} (n-j)
\end{aligned}
$$

Note that

$$\sum_{j=0}^{n-1} (n-j) = n + (n-1) + (n-2) + \ldots + 1 = \sum_{j=1}^{n} j = n(n+1)/2$$

$$\begin{aligned} t_n &= 2 + 2 \cdot \sum_{j=0}^{n-1} (n - j) \\ &= 2 + 2n(n+1)/2 \\ &= 2 + n(n+1) \\ &= \boxed{n^2 + n + 2} \end{aligned}$$

(d) $t_n = 10t_{n-1} - 21t_{n-2} \qquad t_0 = 4 \quad t_1 = 20$

The characteristic equation is

$$x^2 - 10x + 21 = 0$$

There are two root solutions

$$\frac{10 \pm \sqrt{100 - 4 \cdot 21}}{2} = \frac{10 \pm \sqrt{100 - 84}}{2} = \frac{10 \pm \sqrt{16}}{2} = \frac{10 \pm 4}{2} = 14/2 \quad 6/2 = 7 \quad 3$$

The roots are 7 and 3. Now we solve the system of equations

$$\begin{aligned} 4 &= C + D \\ 20 &= 7C + 3D \end{aligned}$$

Multiplying the first by -3 and adding it to the second

$$\begin{aligned} -12 &= -3C - 3D \\ 20 &= 7C + 3D \\ 8 &= 4C \\ C &= 2 \end{aligned}$$

So $C = 2$ and $D = 2$. The formula is $\boxed{t_n = 2 \cdot 7^n + 2 \cdot 3^n}$

2. Use induction to prove your solution to the above recurrences (a) and (c).

Part (a): If $t_n = t_{n-1} + 1 \quad t_0 = 1$, then $t_n = 1 + n$

By induction on n.

- **Base Case**: $t_0 = 1 + 0 = 1$
- **Inductive Step**: $t_n = 1 + n \implies t_{n+1} = 1 + (n+1)$

$$\begin{aligned} t_{n+1} &= t_n + 1 \text{ (by rec.)} \\ &= (1 + n) + 1 \text{ (by IH)} \\ &= 1 + (n+1) \end{aligned}$$

Part(c): If $t_n = t_n - 1 + 2n \quad t_0 = 2$, then $t_n = n^2 + n + 2 = 2 + n(n+1)$

By induction on n

- **Base Case:** $t_0 = 0^2 + 0 + 2 = 2$
- **Inductive Step:** $t_n = n^2 + n + 2 \implies t_{n+1} = (n+1)^2 + (n+1) + 2$

$$\begin{aligned} t_{n+1} &= t_n + 2(n+1) \text{ (by rec.)} \\ &= n^2 + n + 2 + 2(n+1) \text{ (by IH)} \\ &= n^2 + n + 2 + 2n + 2 \\ &= n^2 + 3n + 4 \\ &= (n^2 + 2n + 1) + (n+1) + 2 \\ &= (n+1)^2 + (n+1) + 2 \end{aligned}$$

3. Consider the following sets

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{w, x, y, z\} \\ c &= \{\alpha, \beta\} \end{aligned}$$

(a) Let $f : A \to \mathbb{R}$ where $f(x) = \sqrt{x}$. Is f a well-defined function?

No. Because the square root can be positive or negative. So $f(1) = 1$ or -1.

(b) Let $f : A \to \mathbb{R}$ where $f(x) = |\sqrt{x}|$, where $|\cdot|$ is the absolute value.

i. Is f a well defined function?

Yes. Every element in the domain is defined and maps to a single output in the co-domain.

ii. What is the domain and co-domain of f? Explain.

Domain is $\{1, 2, 3\}$ and the co-domain is $\{1, \sqrt{2}, \sqrt{3}\}$

iii. Is the function one-to-one? Explain.

Yes. No two inputs map to the same output.

iv. Is the function onto? Explain.

No. Not every real number can be generated from the inputs.

(c) Let $g : A \to B$

i. Is it possible for g to be one-to-one? If so, provide a one-to-one mapping.

Yes. $\{(1, w), (2, x), (3, y)\}$

ii. Is it possible for g to *not* be one-to-one? If so, provide a non one-to-one mapping from A to B.

> Yes. $\{(1, w), (2, w), (3, y)\}$

iii. Is it possible for g to be onto? If so, provide a mapping that is onto, and if not, explain why not.

> No. Since the size of the domain is 3, and the size of the co-domain is 4, there is always an element of the co-domain that will not be generated by the function, if it is well defined.

(d) How many well defined functions exist for $h : C \to A$?

> There are three choices for output for each input that could be used in a well-defined function. So there are $9 = 3 \cdot 3$ well defined functions

4. For the following functions, are they well defined, and if so are they either one-to-one and/or onto, or provide a counter example for either.

(a) $f : \mathbb{Z} \to \mathbb{Q} : f(x) = \frac{x}{2}$

> The function is well defined.
>
> - one-to-one: $(\forall x_1, x_2 \in \mathbb{Z})(f(x_1) = f(x_2) \implies x_1 = x_2)$
>
> $$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$
>
> - onto: $(\forall y \in \mathbb{Q})(\exists x \in \mathbb{Z})(f(x) = y)$
>
>   The function is not onto because $\frac{2}{3} \in \mathbb{Q}$ but there does not exist an $x \in \mathbb{Z}$ that would generate that.

(b) $g : \mathbb{Z} \times (\mathbb{Z} - \{0\}) \to \mathbb{Q} : g(a, b) = \frac{a}{b}$

> The function is well defined.
>
> - one-to-one: $(\forall (a_1, b_1), (a_2, b_2) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}))(f(a_1, b_1) = f(a_2, b_2) \implies (a_1, b_1) = (a_2, b_2)$
>
>   The functions is not one to one because $g(4, 2) = 2$ and $g(2, 1) = 2$ but $(4, 2) \not= (2, 1)$
> - onto: For every $y \in \mathbb{Q}$, $y = w/z$ where w and z are integers in reduce formed. Let $a = w$ and $z = b$, then $g(a, b) = y$. Thus, the function is onto.
>
>   Put another way, there always exists ratio to represent y by that can be used as an input to the function g to produce it.

5. Show that the cardinality of $\mathbb{Z} - \{0\}$ is countable.

> To be countable, we need to describe a one-to-one correspondence function with Z^+. Consider the function f:
>
> $$f : \mathbb{Z}^+ \to (\mathbb{Z} - \{0\}), f(x) = \begin{cases} \lfloor x/2 \rfloor + 1 & \text{if } x \text{ is odd} \\ -1 \cdot x/2 & \text{if } x \text{ is even} \end{cases}$$

The function is one-to-one and onto, and thus $\mathbb{Z} - \{0\}$ is countable.

6. Prove the following are equivalence relation, or find a counter example.
   (a) $(\forall a, b \in \mathbb{R}^+)(a\,R\,b \iff 0 < ab \leq 1)$

   - Reflexive: It is not reflexive because not all values are related to themselves. For example, if $a = 2$ would mean than $2^2 = 4 \not\leq 1$
   - Symmetric: $a\,R\,b \implies b\,R\,a$. If $ab \leq 1$, and by commutativity of amplification, so is $ba \leq 1$. It is reflexive.
   - Transitive: $a\,R\,b \wedge b\,R\,c \implies a\,R\,c$. However, 2 and 0.5 as $2 \cdot 0.5 = 1$ are in the relation, and so is 0.5 and 1 because $0.5 \cdot 1 = 0.5$, but 2 and 1 are not in the relation as $2 \cdot 1 = 2$. The relation is not transitive.

   (b) $(\forall a, b \in \mathbb{Z}^+)(a\,S\,b \iff a \mid b)$

   - Reflexive: $a\,S\,a$, then $a|a$, which is true as a value always divides itself.
   - Symmetric: $a\,S\,b \implies b\,S\,a$. Then $a \mid b$, but that does not imply that $b \mid a$. For example $4 \mid 8$ but $8 \nmid 4$
   - Transitive: $a\,R\,b \wedge b\,R\,c \implies a\,R\,c$. So $a \mid b$ and $b \mid c$. That means $\exists k$, such that $ak = b$ and $\exists k'$ such that $bk' = c$. Then $akk' = c$, which implies $a \mid c$. It is transitive.

7. Are any of the above relations partial-order relations?

Yes. The S is a partial order because it is anti-symmetric.

If $a \mid b$ and $b \mid a$, then $a = b$ is true. From the premise, there exists a k such that $ak = b$ and a k' such that $bk' = a$. Substitution provides that $bkk' = b$ so $kk' = 1$ and $k = 1$ and $k' = 1$. Then $a = b$.

The other relation cannot be a partial order because it is not reflexive.

8. Consider the partial order relation over sets A and B

$$(\forall A, B \in \mathcal{P}(\{1, 2, 3\}))({}_A R_B \iff A \subseteq B)$$

prove that it is not a total order relation.

Consider that both $\{2, 3\}$ and $\{1\}$ are elements in the power set, but $\{2, 3\} \not\subseteq \{1\}$ and $\{1\} \not\subseteq \{2, 3\}$. Thus it is not a total ordering.

9. Prove that the following modular equivalence is an equivalence relation:

$$(\forall a, b \in \mathbb{Z})(a\,R\,b \iff a \equiv b \pmod 5)$$

- Reflexive: $a\,R\,a$ implies that $a \equiv_5 a$, or $5|a-a$, which is true because $5|0$.
- Symmetric: $a\,R\,b \implies b\,R\,a$. The fact that $a\,R\,b$ means $a \equiv_5 b$ and so $5|b-a$. That means there exists a k such that $5k = b - a$. Then $a - b = -5k = 5(-k)$ so $5|a-b$ and $b \equiv_5 a$.
- Transitive: $a\,R\,b \wedge b\,R\,c \implies a\,R\,c$. From the premise we can learn that there exists a k and k' such that $5k = b - a$ and $5k' = c - b$. Then $b = c - 5k'$, and so $5k = c - 5k' - a$ or $5(k + k') = c - a$, thus $5|c - a$ and $a\,R\,c$.

10. Find the multiplicative inverse modulo 11 to the following values

    (a) 3

    $3 \cdot 4 = 12 \mod 11 = 1$

    (b) 4

    $4 \cdot 3 = 12 \mod 11 = 1$

    (c) 5

    $5 \cdot 9 = 45 \mod 11 = 1$

11. Consider choose passwords with only values 1-5 and A-E, there are ten total

    (a) How many 4-length password, without repetitions exist?

    $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

    (b) If you were to choose a random 4-length password without repetitions, what is the likelihood that it begins with a 1 and a 5?

    There are There are $8 \cdot 7 = 56$ total passwords that start with a 1 and end with a 5. so the likelihood is $56/5040 = 0.1111 = 11.11\%$

    (c) How many 4-length passwords, with repetitions, exists that contain at least a 1 and an A

    Using inclusion/exclusion, we have

    $$N(A \cap B) = N(A) + N(B) - N(A \cup B)$$

    where A is at least a 1 and B is at least an A.

    For $N(A) = N(B) = 10^4 - 9^4$.

    And $N(A \cup B)$ is at least 1 or at least an A, whose complement is no 1 and no A, of which there are

    $$8^4$$

    of those. Thus $N(A \cup B) = 10^4 - 8^4$.

    The total number is then:

    $$10^4 - 9^4 + 10^4 - 9^4 - 10^4 + 8^4 = 10^4 - 2 \cdot 9^4 + 8^4$$

> There are 10^4 possible passwords, of those 8^4 do not contain a ! or an A. So $10^4 - 8^4$ must contain at least a 1 and an A.

(d) How many 4-length passwords, with repetition, exist that start with an 1 and end with an A, and have at least a 3 as one of the other two digits?

> Consider that there are 10^2 passwords that start with a 1 and end with an A. Of those, 9^2 do not have a 3 as one of the middle digits. So $10^2 - 9^2$ start with 1 and end with an A but at least have a 3.

12. Consider a bucket of 5 balls: 3 balls are red and 2 balls are blue. You draw 3 balls at random.

(a) How many ways can 3 balls be drawn?

> There are $\binom{5}{3} = 10$ ways to choose 3 balls from 5.

(b) What is the likelihood of all 3 balls being red?

> There are $\binom{5}{3} = 10$ ways to choose 3 balls from 5, and only $\binom{3}{3} = 1$ way to choose all red. So 1/10.

(c) What is the likelihood of 2 red balls and 1 blue ball

> There are $\binom{5}{3} = 10$ ways to choose 3 balls from 5, and only $\binom{3}{2} = 3$ to choose 2 red and $\binom{2}{1} = 2$ ways to choose one blue, for a total of 6 ways to choose 2 red and one blue. The Likelihood is 6/10.

(d) If you were told that the first ball is blue, what is the likelihood the remaining two balls are red?

> The probability of drawing at least one blue ball is the complement of drawing no blue balls. The number of ways to draw no blue balls is 1. So 9 ways to draw at least one blue ball. There are 6 ways (from before) to draw two red and one blue, so the conditional probability is $\frac{6/10}{9/10} = \frac{6}{9}$ of drawing two reds given the first ball is blue.