Lec 01: Set Notation

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What is a set?

Definition

A set is a well defined collection of objects that are described as members or elements of the set.

Example

The set of the first 5 prime numbers

$$A = \{2, 3, 5, 7, 11\}$$

The numbers 5 is a member of the set A, or $5 \in A$.

Example

The set of odd numbers between 11 and 17, inclusive.

$$B = \{11, 13, 15, 17\}$$

The number 12 is *not* an element of the Set B, or $12 \notin A$.

Set Roster Notation

The set is described by using braces { } with all elements specified.

Example

- $\{red, blue, orange\}$
- $\{1, \{2, 3\}\}$

We can use ellipses "..." (read "as so forth") to indicate sets that continue on infinitely.

Example

- $\{0, 1, 2, \ldots\}$
- $\{\ldots, -3, -2, -1\}$

Set Equality

Definition

Two sets A and B are equal (written as $A = B$) when they contain the same elements.

Is A equal to B?

- (1) $A = \{1, 2, 3\}$ $B = \{1, 2, 3\}$
  - yes, $A = B$
- (2) $A = \{1, 2, 3\}$ $B = \{2, 1, 3\}$
  - yes, $A = B$ even though elements presented in different orders
- (3) $A = \{1, 2, 3\}$ $B = \{3, 3, 1, 2, 2, 3, 3\}$
  - yes, $A = B$ even with multiple repetitions.
- (4) $A = \{1, 2, 3\}$ $B = \{1, 1, 3\}$
  - No, $A \neq B$ because $2 \in A$ but $2 \notin B$

Common Numeric Sets

- $\mathbb{R}$: The set of real numbers
- $\mathbb{Z}$: The set of integers
- $\mathbb{Q}$: The set of rational numbers
- $\mathbb{N}$: The set of natural numbers
- $\mathbb{C}$: The set of complex numbers

Set Builder Notation

Set Builder Notation

Describe a set where some condition is met.

$$\{x \in S \mid P(x)\}$$

"The set of all elements x in S such that some property/proposition $P(x)$ is true"

Example

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$$

but sometimes we also write it this way

$$\mathbb{Z}^+ = \{x \mid x \in \mathbb{Z} \text{ and } x > 0\}$$

Exercises

Describe all positive even numbers using set-builder notation.

Describe $\mathbb{Q}$ (the rational numbers) using set-builder notation.

Describe $\{x \in \mathbb{Z} \mid x = (-1)^k \text{ where } k \in \mathbb{Z}^+\}$ using set roster notation.

Cardinality

Definition

The size of the set, or its cardinality, is the number of elements in the set.

Example

For the set $A = \{5, 4, 22\}$ the cardinality of A is 3, written $|A| = 3$

Definition

If a set has cardinality of 0, or $|B| = 0$, then we describe it as the empty set and denote it with special symbol $\emptyset$ or sometimes written simply as $\{\}$.

Exercises

Can you construct an argument to show that $\{\} = \emptyset$ based on the equality rule from before?

Recall that "two sets A and B are equal when they contain the same elements"

Is $\{\{\}\} = \emptyset$?

What is $|\{\{\}, 3, \emptyset, \{1, 2, 3\}\}|$?

If two sets A and B have he same cardinality, $|A| = |B|$, is it the case that $A = B$?

If two sets A and B are equal, $A = B$, is it the case that they have the same cardinality, $|A| = |B|$?

Subsets

Definition

For two sets A and B, A is a subset of B (written $A \subseteq B$) if for all elements $x \in A$ then $x \in B$.

Example

For the set $A = \{9, 7, 22\}$ and $B = \{9, 22, 18, 42, 7\}$, $A \subseteq B$ (or $B \supseteq A$) because all the elements of A are also in B, that is $9 \in B$, $7 \in B$ and $22 \in B$.

Definition

A proper subset (written $A \subset B$) is a subset whereby $A \subseteq B$ but there exists at least on element $x \in B$ such that $x \notin A$

Exercises

For the set $A = \{42, 18, 3, 22\}$ and $B = \{22, 3, 42, 18\}$,
is $A \subset B$?
Is $A \subseteq B$?

For two sets, C and D, where $C = D$, is $C \subseteq D$?

For two sets E and F, if $E \subseteq F$ and $E \supseteq F$, then it must be the case that $E = F$. Why?

Is $\emptyset \subset \{5, 12, 2, 19\}$?

Is $\emptyset \subseteq \emptyset$?

Union and Intersection

Definition

The union of two sets A and B, written $A \cup B$, is the set that contains all elements in A *or* B.

Definition

The intersection of two sets A and B, written $A \cap B$, is the set that contains all elements in A *and* B.

Example

If $A = \{12, 7, 19, 5\}$ and $B = \{12, 3, 22, 42, 5\}$ then

$A \cup B = \{12, 7, 19, 5, 3, 22, 42\}$ and
$A \cap B = \{12, 5\}$

Complements or Difference

Definition

The difference between set B and A (or the relative complement of A in B), written $B \backslash A$ or $B - A$, is the set of elements in B that are not in A.

Example

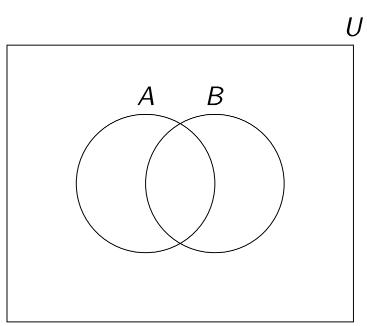
If $A = \{12, 7, 19, 5\}$ and $B = \{12, 3, 22, 42, 5\}$ then

$A \backslash B = A - B = \{7, 19, 5\}$ and
$B \backslash A = B - A = \{3, 22, 42\}$

Definition

The compliment of set A, written A^c is the set of all elements in the universe U from which sets are populated that are not in A.

Set Operators as Venn Diagrams



Exercises

For any set A, what is
$A \cup \emptyset$?
$A \cap \emptyset$?
$A \cup A$?
$A \cap A$?

Write the set builder notation equivalent for each operation, assuming a universal set U.
$A \cup B$
$A \cap B$
$B - A$
A^c

Interval Notation

It's common to want to describe an interval within a numeric set, such as $\mathbb{Z}$ or $\mathbb{R}$. This is easily done using the following notation:

$$\{x \in \mathbb{R} \mid a < x < b\}$$

In plain language, this is the set of real numbers between a and b.

However, this is cumbersome, so we have the following shorthand.

$$\begin{array}{ll} (a,b) = \{x \in \mathbb{R} \mid a < x < b\} & [a,b] = \{x \in \mathbb{R} | a \leq x \leq b\} \\ (a,b] = \{x \in \mathbb{R} \mid a < x \leq b\} & [a,b) = \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a,\infty) = \{x \in \mathbb{R} \mid x > a\} & [a,\infty) = \{x \in \mathbb{R} \mid x \geq a\} \\ (-\infty,a) = \{x \in \mathbb{R} \mid x < a\} & (-\infty,a] = \{x \in \mathbb{R} \mid x \leq a\} \end{array}$$

Power Sets

Definition

The power set of A, written $\mathcal{P}(A)$, is the set of all subsets of A.

Example

The power set of $A = \{1,2,3\}$ is

$$\begin{aligned} \mathcal{P}(A) = \{&\emptyset, \\ &\{1\},\{1,2\},\{1,3\},\{1,2,3\}, \\ &\{2\},\{2,3\}, \\ &\{3\}\} \end{aligned}$$

The cadinality of a power set, $|\mathcal{P}(A)|$ is a power of 2, namely, $|\mathcal{P}(A)| = 2^{|A|}$

Why?

The empty set $\emptyset$ is a member of any power set, why?

Because the size of the power set is always a power of 2, we also denote the power set of a set A as 2^A

Example

$$\mathcal{P}(\{a,b\}) = 2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

Ordered Pairs

Definition

An ordered pair of two elements a and b, written (a,b) is a grouping of a and b where a is the *first element* and b is the *second element*.

Definition

Two ordered pairs are equal, written $(a,b) = (c,d)$, if and only if $a = b$ and $c = d$.

Example

An ordered pair is used to define the Cartesian plane, where the first element is the x component and the second element is the y component.

Two points in the plain are only equal when they have the same x and y

Generalizing ordered pairs into tuples

We can define ordered collections of more than two elements by combining ordered pairs. For example, an ordered collection of three elements and four elements can be written:

$$x = (a,(b,c)) \quad y = (a,(b,(c,d)))$$

As this is cumbersome, we can combine these elements together to form a tuple, denoted like so

$$x = (a,b,c) \quad y = (a,b,c,d)$$

A tuple has the same properties as an ordered pair, where each element of the two tuples must be equal for the tuples to be equal. For example,

$$(10,12,3,5) \neq (10,2,3,5)$$

Cartesian Products (or Cross Products)

Definition

The Cartesian product (or cross product) of two sets A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$.

Example

We can denote the cross product of two sets using set builder notation:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

We can further generalize the result for multiple products,

$$A_1 \times A_2 \times A_3 \times \ldots = \{(a_1,a_2,a_3,\ldots) \mid a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots\}$$

Exercises

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$, what is:
$A \times B$
$B \times A$
$A \times A$
$B \times B$

Let $C = \{p, q\}$, what is $2^C \times B$?