

## Lec 09: Recurrence Relations

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### Exercises

Find the 4th term ( $a_4$  and  $c_4$ ) of the following recurrence relations

$$a_k = 2a_{k-1} + 5$$

$$a_1 = 4$$

$$c_k = c_{k-1} + k \cdot c_{k-2} + 1$$

$$c_0 = 1 \quad c_1 = 2$$

## Recurrence Relations

### Definition

A **recurrence relation** for a sequence  $a_0, a_1, a_2, \dots$  is a formula that relates each term  $a_k$  to  $i$ -number of its predecessors  $a_{k-1}, a_{k-2}, \dots, a_{k-i}$ , where  $k - i \geq 0$ .

The **initial condition** for a recurrence relation specifies the values of  $a_0, a_1, a_2, \dots, a_{i-1}$  needed to start evaluation.

### Example

Recurrence relation for the sum of the positive integers:

$$n_1 = 1 \qquad n_k = n_{k-1} + 1$$

Any positive integer is defined as one more than the previous positive integer. For example  $n_4$  is

$$n_4 = n_3 + 1 = (n_2 + 1) + 1 = ((n_1 + 1) + 1) + 1 = ((1 + 1) + 1) + 1 = 4$$

## Tower of Hanoi

*On the steps of the altar in the temple of Benares, for many, many years Brahminshave been moving a tower of 64 golden disks from one pole to another; one by one, neverplacing a larger on top of a smaller. When all the disks have been transferred the Towerand the Brahmins will fall, and it will be the end of the world.*

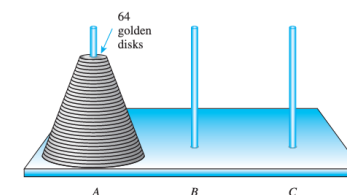
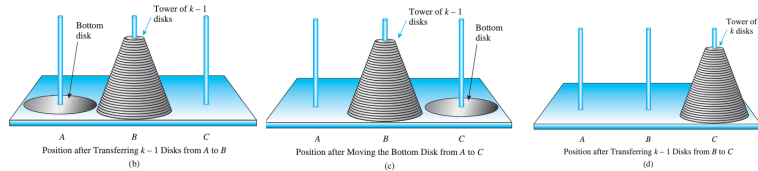


Figure 5.6.2

**When will the world end?** — Assume moving one disk takes 1 second.

## Thinking about the problem recursively

Let  $M_k$  be the number of disk-moves to get  $k$  disks from one pole to another.

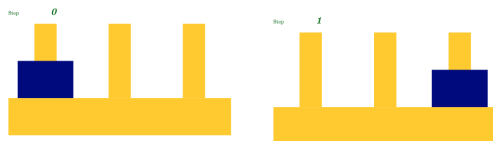


The cost to move  $k$  disks ( $M_k$ ) disks from pole A to pole C equals:

- Cost of moving  $k - 1$  disks from A to B (cost is  $M_{k-1}$ )
- Cost of moving Bottom Disk from A to C (cost is 1)
- Cost of moving  $k - 1$  disks from B to C (cost is  $M_{k-1}$ )

## The base case: $M_0$

With one disk, move it directly from A to C.



$$M_0 = 1$$

## Recurrence relation for Tower of Hanoi (1)

$$M_k = 2M_{k-1} + 1$$

$$M_k = 2(2M_{k-2} + 1) + 1$$

$$M_k = 2(2(2M_{k-3} + 1) + 1) + 1$$

$$M_k = 2(2(2(\dots 2(2M_0 + 1) + 1 \dots) + 1) + 1) + 1$$

## Recurrence relation for Tower of Hanoi (2)

The number of moves  $M_k$  for  $k$  disks is defined as:

$$M_k = 2M_{k-1} + 1$$

$$M_1 = 1$$

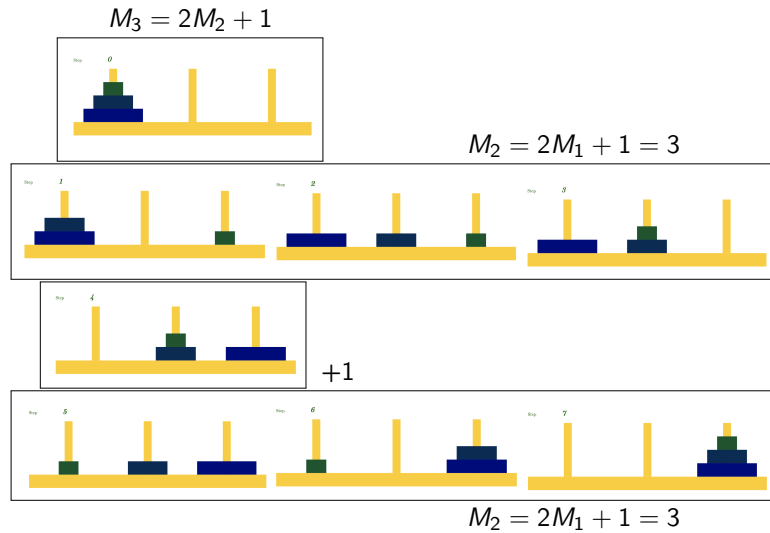
### Example

Number of moves for a tower with 3 disks?:

$$M_3 = 2M_2 + 1 = 2(2M_1 + 1) + 1 = 2(2(1) + 1) + 1 = 7$$

How many moves for a tower with 4 disks?

## Three-disk Hanoi Tower visualized



## When does the world end?

Let's assume that each disk can be moved in 1-second, we can write a short recursive program to calculate  $M_k$ :

```
#M_n
def hanoi(n):
    #M_1 = 1
    if n == 1: return 1

    #M_k = 2*M_{k-1}+1
    else: return 2*hanoi(n-1) + 1
```

$\text{hanoi}(64) = 18446744073709551615$  (seconds)  
 $\cong 584.5 \times 10^9$  (years)

## Solving a recurrence relation

Consider the following recurrence relation:

$$a_k = 2 + a_{k-1}$$

$$a_0 = 1$$

What if we wanted to find a formula for  $a_k$  **without recurrences**?

## Routine for solving a recurrence relation

- 1 Write down the recurrence relation.
- 2 Expand the recurrence relation for several lines. Note how many times you do so, and don't reduce too much.
- 3 Determine what the  $i$ 'th line would be.
- 4 Solve for what value of  $i$  would reach the base case
- 5 In your equation from (3), replace  $i$  with the solution from (4).

### (1-3) Expand for several lines, solve for $i$

$$\begin{aligned} a_k &= 2 + a_{k-1} && \text{step 1} \\ &= 2 + 2 + a_{k-2} && \text{step 2} \\ &= 2 + 2 + 2 + a_{k-3} && \text{step 3} \\ &= 2 \cdot 3 + a_{k-3} \\ &\vdots \\ &= 2(i) + a_{k-i} && \text{step } i \end{aligned}$$

### (4-5) Determine base case, and replace

On the  $i$ -th step the formula is:

$$a_k = 2(i-1) + a_{k-(i-1)}$$

Base case is  $k=0$ , so when does  $k-i=1$ ? When  $i=k$ .

$$\begin{aligned} a_k &= 2i + a_{k-i} \\ &= 2k + a_{k-k} \\ &= 2k + a_0 \\ &= 2k + 1 \end{aligned}$$

### Exercise

Solve the following recurrence relations

$$\begin{aligned} t_k &= t_{k-1} + 9 \\ t_0 &= 11 \end{aligned}$$

$$\begin{aligned} s_n &= n + s_{n-1} \\ s_0 &= 0 \end{aligned}$$

### Exercise (cont)

Prove your solution to the recurrence relations, using induction.

If  $t_k = t_{k-1} + 9$  and  $t_0 = 11$ , then  $t_k = 11 + 9k$

If  $s_n = n + s_{n-1}$  and  $s_0 = 0$ , then  $s_n = n(n+1)/2$

## Solution for Recurrence of Tower of Hanoi (1)

Recall that the formula for the Tower of Hanoi is

$$M_k = 2M_{k-1} + 1 \quad M_1 = 1$$

$$\begin{aligned} M_k &= 2M_{k-1} + 1 && \text{step 1} \\ &= 2(2M_{k-2} + 1) + 1 && \text{step 2} \\ &= 4M_{k-2} + 2 + 1 \\ &= 4(2M_{k-3} + 1) + 2 + 1 && \text{step 3} \\ &= 8M_{k-3} + 4 + 2 + 1 \\ &\vdots \\ &= 2^i M_{k-i} + 2^{i-1} + 2^{i-2} + \dots + 2^0 && \text{let } i = 3 \\ &= 2^i M_{k-i} + \sum_{j=0}^{i-1} 2^j && \text{step } i \end{aligned}$$

## Solution for Tower of Hanoi (2)

The base case is when  $k = 1$ , so when  $i = k - 1$ , we reach a base case:

$$\begin{aligned} &= 2^i M_{k-i} + \sum_{j=0}^{i-1} 2^j && i = k - 1 \\ &= 2^{k-1} M_{k-(k-1)} + \sum_{j=0}^{(k-1)-1} 2^j \\ &= 2^{k-1} M_1 + \sum_{j=0}^{k-2} 2^j && M_1 = 1 \\ &= \underbrace{2^{k-1} + \sum_{j=0}^{k-2} 2^j}_{\text{Same as summing to } k-1} = \underbrace{\sum_{j=0}^{k-1} 2^j}_{\text{Geometric Sum}} \\ &= \frac{1 - 2^k}{1 - 2} = 2^k - 1 \end{aligned}$$

## Proving the Tower of Hanoi

With a solution, we can prove it using induction:

### Theorem

The  $k$ -th step of the recurrence relation  $M_k = 2M_{k-1} + 1$ , where  $M_1 = 1$ , for all  $k > 1$ , is  $2^k - 1$

### Proof.

By induction on  $k$ ,

**Base Case:  $k=2$ :**  $M_2 = 2(1) + 1 = 3 = 2^2 - 1$

**Inductive step:** Assume  $M_k = 2^k - 1$  for all, show that  $M_{k+1} = 2^{k+1} - 1$ .

Note that,  $M_{k+1} = 2M_k + 1$  and by the induction hypothesis,  $M_k = 2^k - 1$ . Substituting in, we have

$M_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$ , showing our result.  $\square$

## (extra) Exercises

Solve the following recurrence relations

$$t_k = 2k + t_{k-1} \quad t_0 = 2$$

$$t_k = 3t_{k-1} + 2 \quad t_1 = 3$$