Lec 10:
Recurrence Relations II

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Geometric recurrence

A geometric recurrence of the form

$$a_n = 2 \cdot a_{n-1}$$
$$a_0 = 3$$

can also be describes as a first-order, linear homogeneous recurrence with constant coefficients because we only refer back one iteration of the recurrence (first order), n does not factor into formula of the recurrence (homogeneous), and the coefficients is constant (which is 2 in this case, but could be any number).

Exercise: find a solution for a_n.

Solving geometric recurrence using expansion

$$a_n = 2 \cdot a_{n-1} \qquad \text{Step 1}$$
$$a_n = 2 \cdot 2 \cdot a_{n-2} \qquad \text{Step 2}$$
$$a_n = 2 \cdot 2 \cdot 2 \cdot a_{n-3} \qquad \text{Step 3}$$
$$\vdots$$
$$a_n = 2^i a_{n-i} \qquad \text{Step i}$$

$i = n$ to reach a_0

$$a_n = 2^n \cdot a_0$$
$$a_n = 3 \cdot 2^n$$

Solving recurrence using characteristic equation

Another way to understand a recurrence is based on the properties of the sequences that "solve" it. In particular, suppose there exists some sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots$$

for some real number t that is a *solution* to the recurrence at each iteration; that is, the recurrence relation $a_n = 2 \cdot a_{n-1}$ generate that sequence.

We just don't know what t is, yet.

Rewriting the recurrence in terms of t

If the sequence $1, t, t^2, t^3, \dots, t^n, \dots$ is generated by the recurrence $a_n = 2 \cdot a_{n-1}$, then

$$\begin{aligned} a_0 &= t^0 \\ a_1 &= t^1 \\ a_2 &= t^2 \\ &\vdots \\ a_{n-1} &= t^{n-1} \\ a_n &= t^n \end{aligned}$$

When $n \geq 1$, we can rewrite a_n in terms of t.

$$t^n = 2 \cdot t^{n-1}$$

Solving for t

Solving for t is a matter of algebraic manipulation

$$\begin{aligned} t^n &= 2 \cdot t^{n-1} \\ t^n - 2 \cdot t^{n-1} &= 0 \\ \frac{t^n - 2 \cdot t^{n-1}}{t^{n-1}} &= \frac{0}{t^{n-1}} \\ t - 2 &= 0 \\ t &= 2 \end{aligned}$$

We can then define the sequence r as being generated by the recurrence $a_n = 2 \cdot a_{n-1}$, where r is

$$r = \underbrace{1}_{t^0}, \underbrace{\overbrace{2}^{2\cdot 1}}_{t^1}, \underbrace{\overbrace{4}^{2\cdot 2}}_{t^2}, \underbrace{\overbrace{8}^{4\cdot 2}}_{t^3}, \dots, \underbrace{\overbrace{n}^{2\cdot r_{n-1}}}_{t^n}$$

and $r_n = 2^n$. But r does not meet the initial conditions of the sequence a we are trying to solve.

Solving for initial conditions

(We'll prove the second-order form of this lemma later)

Lemma: First-Order ...

If $r_0, r_1, \dots$ satisfies a first-order, homogeneous recurrence relation with constant coefficients, for constant A where $r_n = A \cdot r_{n-1}$, and if the sequence $a_0, a_1, \dots$ also satisfies the recurrence, then

$$a_n = C \cdot r_n$$

for some constant C.

Using the lemma, we have $r_n = 2^n$, so $r_0 = 2^0 = 1$. Then we can solve for C where $a_0 = 3$.

$$a_0 = C \cdot 1 = C = 3$$

And the formula is $\boxed{a_n = 3 \cdot 2^n}$.

Second Order, Linear Homogeneous Recurrences

Definition

A second-order, linear homogeneous recurrence relation with constant coefficient is a recurrence of the form

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

For all $k \geq b$, where b is a fixed integer, and A and B are constant (real) numbers.

Example

The Fibonacci are such a recurrence where $r = 1$ and $A = 1$ and $B = 1$

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ f_1 &= 1 \\ f_2 &= 1 \end{aligned}$$

Solving an example second-order recurrence

Consider the recurrence relation below

$$a_n = 5a_{n-1} - 6a_{n-2}$$
$$a_0 = 7$$
$$a_1 = 16$$

Again, we try and find a sequence $1, t, t^2, \ldots, t^n$ that is generated by this relation. Using the same reasoning, we have the following formula for solving for t

$$t^n = 5 \cdot t^{n-1} - 6 \cdot t^{n-2}$$

Reducing to a Quadratic Equation

Solving for t is the same as solving a quadratic

$$\begin{aligned} t^n &= 5t^{n-1} - 6t^{n-2} \\ t^n - 5t^{n-1} + 6t^{n-2} &= 0 \\ (1/t^{n-2}) \cdot (t^n - 5t^{n-1} + 6t^{n-2}) &= 0 \cdot (1/t^{n-2}) \\ \underbrace{\boxed{t^2 - 5t + 6}}_{\text{Characteristic Equation}} &= 0 \end{aligned}$$

Applying the quadratic equation

$$\frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 6 \cdot 1}}{2 \cdot 1} = \frac{5 \pm 1}{2} = 2 \text{ and } 3$$

Both $t = 2$ and $t = 3$ are two real solutions to the quadratic equation.

Two sequences solution

For the recurrence, $a_n = 5a_{n-1} - 6a_{n-2}$, we have found two sequences that would be generated by the recurrence relation:

$$r = 1, 2, \overbrace{2^2}^{5\cdot2-6\cdot1}, \overbrace{2^3}^{5\cdot4-6\cdot2}, \ldots, \overbrace{2^n}^{5\cdot2^{n-1}-6\cdot2^{n-2}}$$
$$s = 1, 3, \underbrace{3^2}_{5\cdot3-6\cdot1}, \underbrace{3^3}_{5\cdot9-6\cdot3}, \ldots, \underbrace{3^n}_{5\cdot3^{n-1}-6\cdot3^{n-2}}$$

Like before, how do we find a unified formula for a_n that take into account the initial conditions a_0 and a_1.

Combining solution sequences

Lemma: Second-Order ...

If $r_0, r_1, \ldots$ and $s_0, s_1, \ldots$ are sequences satisfying the same second-order, linear homogeneous recurrence relation for real constants A and B, that is

$$r_k = A \cdot r_{k-1} + B \cdot r_{k-2} \text{ and } s_k = A \cdot s_{k-1} + B \cdot s_{k-1}.$$

and if the sequence $a_0, a_1, \ldots$ is defined as

$$a_n = C \cdot r_n + D \cdot s_n$$

for real constants C and D, **then** it also satisfies the same relation, namely,

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

Proof of Lemma

Proof.

To prove this, we must ultimately show that the following equality is true

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

when we assume that sequence $r_0, r_1 \ldots$ and $s_0, s_1, \ldots$ also satisfy the relation, and $a_n = C \cdot r_n + D \cdot s_n$. We proceed by algebraic manipulation:

$$\begin{aligned} A \cdot a_{k-1} + B \cdot a_{k-2} &= A(C \cdot r_{k-1} + D \cdot s_{k-1}) + B(C \cdot r_{k-2} + D \cdot s_{k-2}) \\ &= A \cdot C \cdot r_{k-1} + B \cdot C \cdot r_{k-1} + A \cdot D \cdot s_{k-2} + B \cdot D \cdot s_{k-2} \\ &= C(A \cdot r_{k-1} + B \cdot r_{k-1}) + D(A \cdot s_{k-1} + B \cdot s_{k-2}) \\ &= C \cdot r_k + D \cdot s_k \\ &= a_k \end{aligned}$$

□

Solving a linear equation

From the lemma, we know that a sequence $a_0, a_1, \ldots$ also satisfies the recurrence relation, would have a formula for a_n as

$$a_n = C \cdot 2^n + D \cdot 3^n$$

From our initial conditions, we have $a_0 = 7$ and $a_1 = 16$, providing the following system of equations:

$$\begin{aligned} 7 &= C + D \\ 16 &= 2C + 3D \end{aligned}$$

Then $D = 2$ and $C = 5$, and the formula for a_n is:

$$\boxed{a_n = 5 \cdot 2^n + 2 \cdot 3^n}$$

Proof of solution to example (1)

Theorem

The recurrence relation

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} \\ a_0 &= 7 \\ a_1 &= 16 \end{aligned}$$

is defined by the formula $a_n = 5 \cdot 2^n + 2 \cdot 3^n$

Proof

By induction on n, we have two base cases:

$$\begin{aligned} a_0 &= 5 \cdot 2^0 + 2 \cdot 3^0 = 7 \\ a_1 &= 5 \cdot 2^1 + 2 \cdot 3^1 = 16 \end{aligned}$$

Proof of solution to example (2)

Proof (cont.)

In the inductive step, if we assume $a_n = 5 \cdot 2^n + 2 \cdot 3^n$, then we must show $a_{n+1} = 5 \cdot 2^{n+1} + 2 \cdot 3^{n+1}$. By the definition of the recurrence, we can expand a_{n+1}, and then apply the IH.

$$\begin{aligned} a_{n+1} &= 5a_n - 6a_{n-1} \\ a_{n+1} &= 5(5 \cdot 2^n + 2 \cdot 3^n) - 6(5 \cdot 2^{n-1} + 2 \cdot 3^{n-1}) \\ a_{n+1} &= 25 \cdot 2^n + 10 \cdot 3^n - 30 \cdot 2^{n-1} - 12 \cdot 3^{n-1}) \\ a_{n+1} &= 2^{n-1}(25 \cdot 2 - 30) + 3^{n-1}(10 \cdot 3 - 12) \\ a_{n+1} &= 2^{n-1}(20) + 3^{n-1}(18) \\ a_{n+1} &= 2^{n-1}(2 \cdot 2 \cdot 5) + 3^{n-1}(3 \cdot 3 \cdot 2) \\ a_{n+1} &= 5 \cdot 2^{n+1} + 2 \cdot 3^{n+1} \end{aligned}$$

□

Steps for finding a solution to a Second-Order, Linear Homogeneous Recurrence Relation with Constant Coefficients

1. Identify the characteristic equation of the relation from the recurrence
   - $a_n = A \cdot a_{n-1} + B \cdot a_{n-2}$
   - $t^2 - A \cdot t - B = 0$
2. Find the roots using the quadratic formula
   - Does it have two real roots? r and s (continue)
   - Does it have a single root? r (see later)
   - If solutions are not real, then there isn't a formula
3. Rewrite formula in terms of $a_n = C \cdot r^n + D \cdot s^n$
4. Solve the linear equations from initial conditions of a_0 and a_1 to determine C and D
   - $a_0 = C + D$
   - $a_1 = C \cdot r + D \cdot s$

Exercise

Find a solution to the following recurrences

$a_n = a_{n-1} + 2a_{n-2} \qquad a_0 = 8 \quad a_1 = 4$

$a_n = 6a_{n-1} - 8a_{n-2} \qquad a_0 = 12 \quad a_1 = 8$

Prove the formulas using induction

Single Root Solution

If there is only one root to the characteristic equation,

$$t^2 - A \cdot t - B = 0$$

Then there are two sequences, $1, r, r^1, r^2, r^3, \ldots$ and $0, r, 2r^2, 3r^3, \ldots$ in the relation, and

$$a_n = C \cdot r^n + D \cdot n \cdot r^n$$

We can again solve for linear equation with the initial conditions to find C and D.

Single Root Example (1)

To solve the following recurrence

$$a_n = 2a_{n-1} - a_{n-2} \qquad a_0 = 1 \quad a_1 = 4$$

We have the characteristic equation $t^2 - 2t + 1 = 0$, with only one solution when $t = 1$.

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2} = \frac{2 \pm \sqrt{0}}{2} = 1$$

So the sequences are solutions to the recurrence

$$1, 1^2, 1^3, 1^4, \ldots, 1^n = 1, 1, \overbrace{1}^{2\cdot 1 - 1}, \overbrace{1}^{2\cdot 1 - 1}, \ldots, \overbrace{1}^{2a_{n-1} - a_{n-2}}$$

$$0, 1, 2\cdot 1^2, 3\cdot 1^3, 4\cdot 1^4, \ldots, n\cdot 1^n = 0, 1, \underbrace{2}_{2\cdot 1 - 0}, \underbrace{3}_{2\cdot 2 - 1}, \ldots, \underbrace{n}_{2a_{n-1} - a_{n-2}}$$

Single Root Example (2)

a_n can be expressed as

$$a_n = C \cdot 1 + D \cdot n \cdot 1 = C + Dn$$

We now solve for C and D in the following system of equations with $a_0 = 1$ and $a_1 = 1$

$$1 = C + 0 \cdot D$$
$$4 = C + D$$

So $C = 1$ and $D = 3$, and the solution to the recurrence

$$a_n = 2a_{n-1} - a_{n-2} \qquad a_0 = 1 \quad a_1 = 4$$

is

$$a_n = 1 + 3n$$

exercise

Solve the following recurrences

$$a_n = 6a_{n-1} - 9a_{n-2} \qquad a_0 = 4 \quad a_1 = 9$$

And prove your final formula.