

Lec 13: Relations I

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Relations as sets (and subsets)

Moreover, we can define the relation R as a set itself. Again, if R is a relation between two sets A and B and $x \in A$ and $y \in B$, then R can be defined such that:

$$x R y \iff (x, y) \in R$$

Definition

Let A and B be sets. Then a **relation R from A to B** is a subset of $A \times B$. Given pair $(x, y) \in A \times B$, x is related to y by R , written $x R y$, if, and only if, $(x, y) \in R$.

The set A is the **domain** of R , and set B is the **co-domain** of R .

Relations between sets

Relations, like functions, are a way to describe how elements in one set relate to elements in another set.

Example

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$. Then $x \in A$ is related to $y \in B$ if, and only if, $x < y$. Symbolically

$$x R y \iff x < y$$

Where $x R y$ is read "x related to y."

Enumerate all pairs $(x, y) \in A \times B$ that are in the relation?

Reflexive, Symmetric and Transitive

Let R be a relation on a set A :

- R is **reflexive**, if and only if,

$$\forall x \in A, x R x$$

- ▶ All elements in A are related to themselves.

- R is **symmetric** if, and only if,

$$\forall x, y \in A, x R y \implies y R x$$

- ▶ For any x, y in A , if x is related to y , then y is related to x

- R is **transitive**, if, and only if,

$$\forall x, y, z \in A, x R y \wedge y R z \implies x R z$$

- ▶ For any x, y, z in A , if x is related to y and y is related to z , then x is related to z .

Exercise

How would you define (formally) the inverse of

R is **not reflexive** if, and only if, ...

R is **not symmetric** if, and only if, ...

R is **not transitive** if, and only if, ...

Exercise

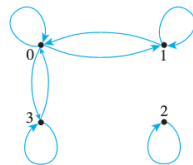
Let $A = \{0, 1, 2, 3\}$ and define the relation R as

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$

Is R reflexive? symmetric? transitive?

Direct graph depiction of relations

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$



- Reflexive requires self loops for all elements in A
- Symmetry requires loops to exist between any related elements
- Transitivity requires closed connections between co-related terms. So there should be an arrow between 3-to-1 and from 1-to-3.

Equivalence Relations

Definition

Let A be a set and a R a relation on A . R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Note that a relation is not an equivalence relation if there exists at least one counterexample of reflexivity, symmetry, or transitivity.

Example equivalence relation: =

Define R as a relation over the reals such that

$\forall x, y \in \mathbb{R}$

$$x R y \iff x = y$$

Two items are in the relation if they are equal: **is this an equivalence class?**

That is, is R reflexive, symmetric and transitive?

Less than relation as equivalence?

Define the relation R over the reals as

$\forall x, y \in \mathbb{R}$

$$x R y \iff x < y$$

Is the less than relation R an equivalent relation?

= as an equivalence relation

Proof.

For R to be **reflexive**, we must show that $\forall x \in \mathbb{R}, x R x$, or equivalently show that $x = x$, which is true for equality, so R is reflexive.

For R to be **symmetric**, we must show that $\forall x, y \in \mathbb{R}, x R y \implies y R x$, or equivalently, $x = y \implies y = x$, which is certainly true for equality of the reals, so R is symmetric.

For R to be **transitive**, we must show that $\forall x, y, z \in \mathbb{R}, x R y \wedge y R z \implies x R z$, or equivalently, that if $x = y$ and $y = z$, then $x = z$, which is also true for equality of the reals, so R is transitive.

Thus R is a equivalence relation. □

Exercises

Are the following equivalence relations? Try to prove your result, and if they are not, which property is violated.

Let R be the relation on \mathbb{Z}^+ such that

$$(\forall n, m \in \mathbb{Z}^+)(n R m \iff n|m)$$

Let S be the relation on \mathbb{R} such that

$$(\forall x, y \in \mathbb{R})(x S y \iff xy \geq 0)$$

Antisymmetry

Definition

Let R be a relation on set A . R is **antisymmetric** if, and only if, for all a and b in A , if $a R b$ and $b R a$ then $a = b$

More formally

$$R \text{ is antisymmetric} \iff (\forall a, b \in A)[(a R b \wedge b R a) \implies a = b]$$

Contrapositive defines **not antisymmetric**

$$R \text{ is not antisymmetric} \iff (\exists a, b \in A)(a R b \wedge b R a \wedge a \neq b)$$

Recall that $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Partial order relation

Definition

A relation R over a set A is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

Antisymmetric ensures that we can create some chain of ordering over elements. For example, for the divides relation $a R b$ if, and only if, $a \mid b$, we can say that $a \preceq b$ if $a R b$, leading to various ordered chains factors, all ending at 1.

$$\begin{aligned} 120 \succcurlyeq 10 \succcurlyeq 5 \succcurlyeq 1 \\ 100 \succcurlyeq 20 \succcurlyeq 4 \succcurlyeq 2 \succcurlyeq 1 \end{aligned}$$

We use the \preceq and \succcurlyeq to indicate “less than or equal” or “greater than or equal” under the relation as not to be confused with the canonical less/greater than (\leq, \geq).

Example antisymmetric relation: divides

Let R be a relation over \mathbb{Z}^+ , and $a R b$ if, and only if, $a \mid b$. R is antisymmetric.

Proof.

Suppose that $a R b$ and $b R a$, we must show that $a = b$. By definition of the relation $a R b$ and $b R a$ implies that

$$\begin{aligned} a R b &\implies a \mid b \implies a = br \text{ for some positive integer } r \\ b R a &\implies b \mid a \implies b = as \text{ for some positive integer } s \end{aligned}$$

Substituting for b in the first formula we have

$$\begin{aligned} a &= br \\ a &= (as)r \\ 1 &= sr \end{aligned}$$

Since s and r are positive integers, they must be 1. Substituting for r we have $a = 1b$ and thus $a = b$. \square

Is the relation R antisymmetric over the positive and negative integers?

Exercise

Let \mathcal{A} be a collection of sets. Show that \subseteq is a partial ordering over \mathcal{A} , that is for sets a and b in \mathcal{A} , $a R b$ if, and only if, $a \subseteq b$.

Total Ordering

In partial ordering, it is possible to find two elements a and b that are not related. For example, in the divides relation, $a \nmid b$ would not be in the relation, and thus a and b could not be ordered.

Definition

A partial order relation R on set A is a **total order relation** if for any two elements a and b either $a R b$ or $b R a$.

Example: The less than relation $a \leq b$ is a total order relation.