Lec 16:
Prob. and Counting II

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Factorials and Permutations

Multiplication Rule and Factorials

The multiplication rule states that we can multiply the set of possible outcomes for a series of choices to get the total number of outcomes.

Example: Assume we have four elements, $\{A, B, C, D\}$, how many different ways can they be arranged (**without** repetition)?

There are 4 choices for the first element, 3 for the second, 2 for the third, and then just 1 for the last.

$$4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

Factorial

Definition

The factorial of a number n, written $n!$, is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

And definition ally, $0! = 1$.

What is

- $4!$
- $5!$
- $\frac{5!}{3!}$

Permutations

Definition

A permutation of a subset of objects is the number of ways those objects can be arranged in a row.

Example

If there are 6 diplomats, and 4 seats around a table, how many different permutations are there for 6 diplomats to sit at the table where two diplomats stand?

$$6 \cdot 5 \cdot 4 \cdot 3 = 6!/2! = 360$$

6 diplomatic in the first seat, 5 can be selected for the second seat, 4 in the third, and 3 in the fourth.

r-Permutation

Definition

An r-permutation of a set of n elements is an ordered selection of r elements take from the set of n elements. The number of r-permutations of a set of n elements is denoted as $P(n, r)$ or $n\,P\,r$, read "n permute r."

If n and r are integers and $1 \leq r \leq n$, the number of r-permutations of a set n elements is given by the formula:

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

or equivalently

$$P(n, r) = \frac{n!}{(n-r)!}$$

Exercise

How many 4-permutations of a set of 7?

How many ways can you arrange 3 letters from a 26 letter alphabet?

Prove that for all integers $n \geq 2$, $P(n, 2) + P(n, 1) = n^2$

Addition, Difference, Unions, and Intersections of Counting/Probability

Example: PINs of multiple lengths

Suppose you want to count the number of 1-, 2-, 3-, and 4-length PINs selected from digits 0-9 without repetition?

Calculation at each length:

- 1-length: $P(10, 1) = 10 = 10$
- 2-length: $P(10, 2) = 10 \cdot 9 = 90$
- 3-length: $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$
- 4-length: $P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

The total PINs of these lengths is the sum, $10 + 90 + 720 + 5040 = 5860$

Addition Rule for Disjoint Sets

Addition Rule

Suppse a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$, Then

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$$

Example: PINS of different lengths are disjoint sets. So to count them, we add up the lengths of each length.

Exercise

Using the addition rule, how many three digit numbers 100-999 are divisible by 5?

Example: Counting PINs without repeated symbols

How many PINs of length 4 (with repetition) **do not** contain a repeated digit?

Consider the following:

- When there are **no repetitions** of digits, there are $P(10, 4) = 5040$ PINs
- When there are **repetitions** of digits, there are $10^4 = 10000$ PINs

If we **subtract** the number of PINs that **do not** have repeated digits from the total number of PINs, we are left with just the PINs that have **at least** on repeated digit.

$$10000 - 5040 = 4060 \text{ PINs with at least one repeated digit}$$

The Difference Rule

The Difference Rule

If A is a finite set, and B is a subset of A, then

$$N(A - B) = N(A) - N(B)$$

The set of PINs with at least one repeated digits is the set-difference between the set of all PINs and the set of PINs without any repeated digits.

Exercise

What is the probability of randomly selecting a 4-digit PIN (with repetitions) that contains at least one 5?

Complements

Probability of a Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A)$$

The probability of selecting a PIN with at least one 5 is the same as the complement of the probability of *not* selecting a PIN with no 5's.

Counting Unions (with overlaps)

How many numbers between 1 and 1000 are divisible by 3 or 5?

We can calculate the number of divisible numbers for both 3 and 5

- $\lfloor 1000/3 \rfloor = 333$
- $\lfloor 1000/5 \rfloor = 200$

But we can not take the sum of the two numbers to answer the question because some numbers are divisible by BOTH 3 and 5.

For a number to be visible by 3 and 5, it is divisible by 15, and $\lfloor 1000/15 \rfloor = 66$ would be counted twice. We can subtract that from the total.

Thus there are $333 + 200 - 66 = 467$ numbers between 1 and 1000 that are divisible by 3 or 5.

Inclusion/Exclusion Rule

The inclusion/Exclusion rule for two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$\begin{aligned} N(A \cup B \cup C) = &N(A) + N(B) + N(C) \\ &- N(A \cap B) - N(B \cap C) - N(A \cap C) \\ &+ N(A \cap B \cap C) \end{aligned}$$

What is another way to define $N(A \cap B)$?

$$N(A \cap B) = N(A) + N(B) - N(A \cup B)$$

Probability of General Unions and Intersections

Recall that a $P(E)$, or the probability of E, is $N(E)/N(S)$, that is, the number of ways an event can occur divided by the size of the sample space.

What if we had two kinds events, A and B:

- Probability of A **or** B is the same as $P(A \cup B)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap b)$
- Probability of A **and** B is the same as $P(A \cap B)$
  - $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Combinations and Poker Hands

Number of Subsets of a Given Size

Consider the set, $\{A, B, C, D\}$, how many subsets of size 2 exist?

- $\{A, B\}$,$\{A, C\}$,$\{A, D\}$
- $\{B, C\}$, $\{B, D\}$
- $\{C, D\}$

This is less than the ways to permute 2 items from the set, in fact half the size.

- AB, BA, CA, AC, AD, DA
- BC, CB, BD, DB
- CD, DC

Calculate the number of subsets of size 3 exist?
How does that compare to the number of permutations of size 3?

Combination

Definition

Let n and r be non-negative integers with $r \leq n$. An r-combination of a set of n elements is a subset of r of the n elements. We write a combination as

$$\binom{n}{r}$$

read "n choose r". It is also denoted as $C(n, r)$ and $n\,C\,r$.

A combination can be computed by dividing the number of permutations by the different arrangements, or $r!$.

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Exercise

If you flipped a coin 5 times, how many different ways can you get 3 heads?

If you flip a coin 5 times, what is the likelihood you get at least 2 head.

If you had a bucket of balls, 5 blue, 3 red, 2 green, and you selected 4 of them, what is the probability of drawing 2 blue, 2 red, and 1 green ball.

Poker Hands

Recall that a deck of cards has 4 suites, each with 13 cards: A, 2, 3, . . . , 10, J, Q, K. A poker hand is a selection of 5 cards selected at random.
How big is the sample space of poker hands?

- $N(\text{poker-hands}) = \binom{52}{5} = 2\,598\,960$

How many poker hands contain at least a pair? Two cards that are the same?

- $P(\text{at least one pair}) = 1 - P(\text{no-pair}) = 1 - N(\text{no-pair})/N(\text{poker-hands})$
- $N(\text{no-pair}) = \overbrace{\binom{13}{5}}^{\text{Choose 5 cards of differnt rank}} \cdot \underbrace{\binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}_{\text{choose suite of each card}} = 1\,317\,888$
- $P(\text{at least one pair}) = 1 - 1\,317\,888/2\,598\,960 = 1 - 0.507 = 0.493 = \boxed{49.3\%}$

Poker Hand Exercise

How many poker hands contain four of a kind?

How many poker hand are in a full house

How many poker hands are a flush? (all the same suite?)