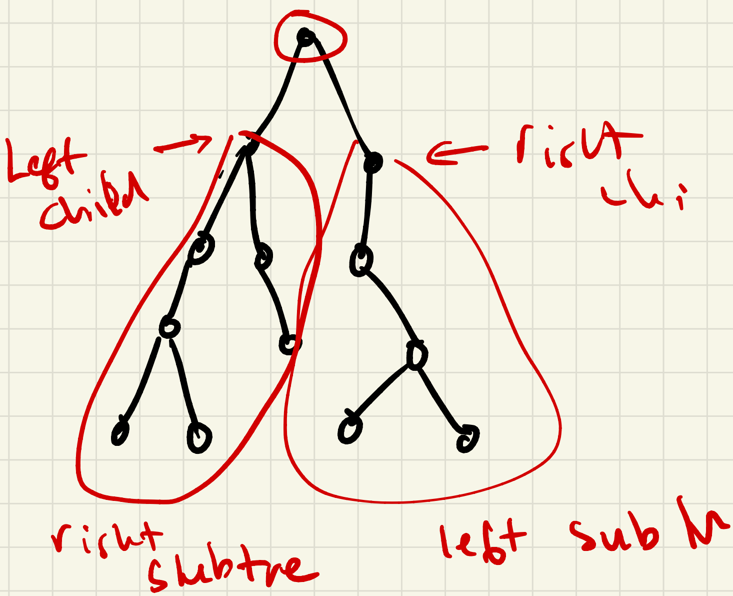
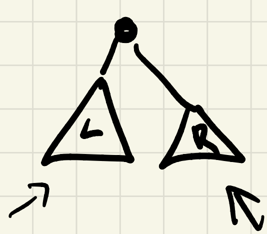
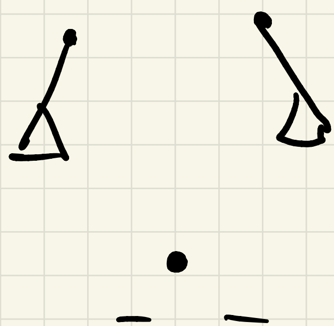
Binary Tree

A rooted tree with a branchy factor of 2







A full binary tree

$2^0 - 1$

$2^1 - 2$

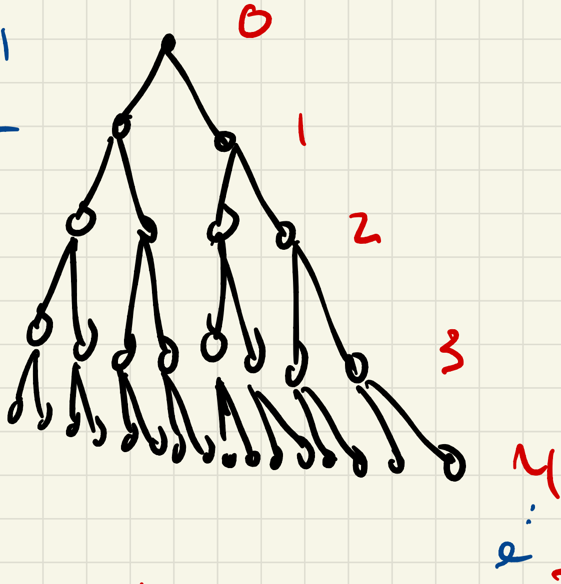
$2^2 - 4$

$2^3 - 8$

$2^4 - 16$

$\vdots \quad 32$

2^ℓ



How many vertices are there?

$$\boxed{\sum_{\ell=0}^{h} 2^\ell}$$

geometric sum!

Geometric sum

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

$$\sum_{\ell=0}^{h} 2^\ell = \frac{1-2^{h+1}}{1-2}$$

$$\boxed{= 2^{h+1} - 1}$$

$h = n-1$

$h+1 = n$

# of vertices in a <u>full</u> binary tree

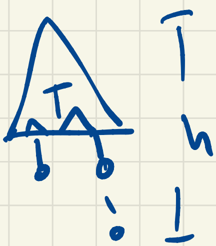
$N(h)$: number of vertices at a height

If binary tree T is not full and has height h then

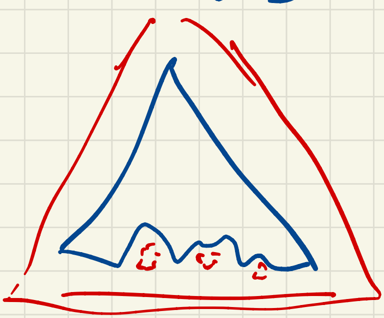
$$N(h) \le 2^{h+1}$$

T' were full

$$N(h') = 2^{h'+1} - 1$$



$h > h'$



$$2^{h+1} - 1$$

$$N(h) < 2^{h+1} - 1$$

$$N(h) \le 2^{h+1}$$

Induction on Binary Trees

$P(0)$, $\forall k \leq n \; P(k) \Rightarrow P(n+1)$

Show

property we are trying to prove

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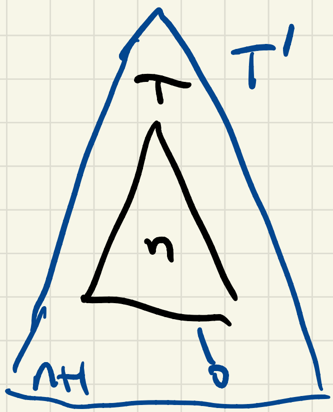
n : number of vertices

$P(1)$ | $P(n) \Rightarrow P(n+1)$

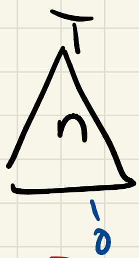
A tree with one vertex

The property hold for a tree T with n vertice $\Rightarrow$ The property hold for a Tree T' with $n+1$ vertices

To Show



remove one vertex



how many ways can I add this vertex!

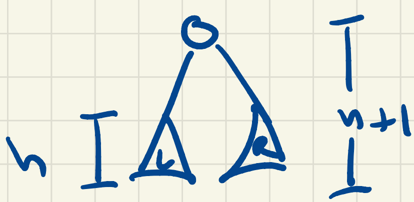
$$\frac{\forall k \le n \quad P(k) \implies P(h+1)}{P(h) \implies P(h+1)}$$

$P(0)$

height 0

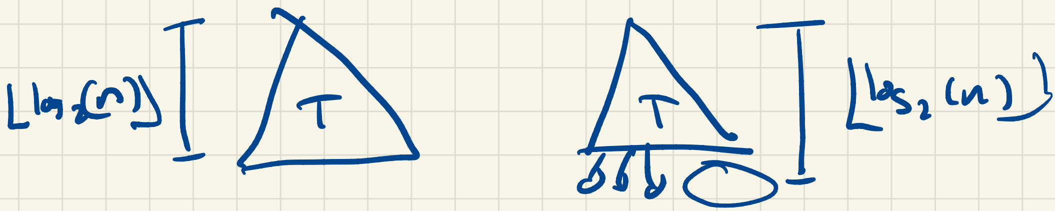
Property holds for tree T with height h
$$\overline{\quad\quad\quad}$$
IH

$\implies$ Property holds for tree T' with height h+1
$$\overline{\quad\quad\quad}$$
TS



look at sub trees, apply the IH and then try to reason the tree with h+1 height

If a full tree, except at the last level, we have $n \geq 1$ vertices, the height of the tree is $\lfloor \log_2 n \rfloor$



$P(1)$ $\quad \bullet \; \} \; h = 0$

$$\log_2 1 = 0$$

$$2^0 = 1$$

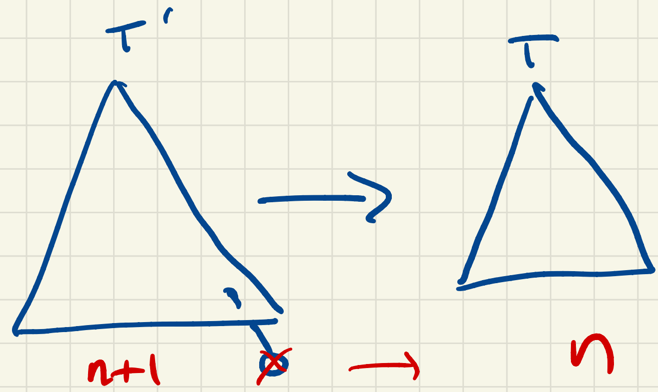
BASE CASE

IH   TS

$P(n) \implies P(n+1)$

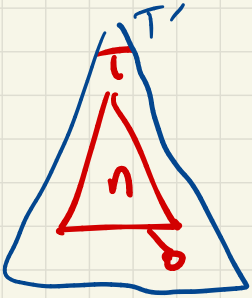
A full binary tree, T, except at last level, has a height of $h = \lfloor \log_2(n) \rfloor$

A full binary tree T', except at last level, has a height of $h' = \lfloor \log_2(n+1) \rfloor$



How can add back that vertex to a tree T with n vertices?

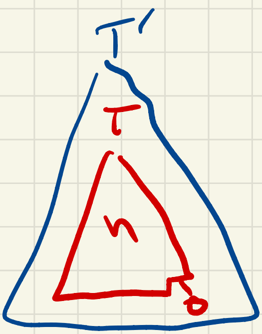
How do use the info from the IH

① 

$\lfloor \log_2(n+1) \rfloor = h+1$

T is a full bin

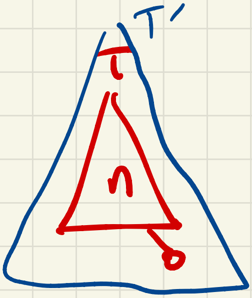
the adding back of one vertex increases the height of T by 1

② 

$\lfloor \log_2(n+1) \rfloor = h$

T is full, except at last level, so adding one more vertex does not increase the height of T

1



$\lfloor \log_2(n+1) \rfloor = h+1$

T is a full bin

the adding back of one vertex increases the height of T by 1

By IH, height of T is h and its a full binary tree

$$n = 2^{h+1} - 1$$

$$+1 \qquad\qquad +1$$

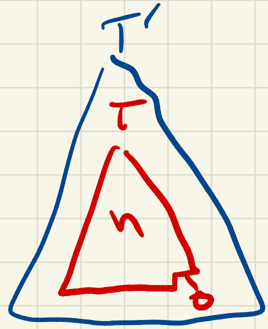
node added back in to reach T' w/ n+1 nodes

$$\boxed{n+1 = 2^{h+1}}$$

$$\log_2(n+1) = \log_2(2^{h+1})$$

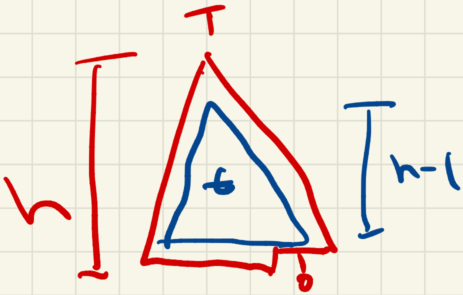
$$\lfloor \log_2(n+1) \rfloor = h+1$$

2



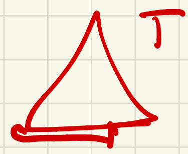
T is full, except at last level, so adding one more vertex does not increase the height of T

$$\lfloor \log_2(n+1) \rfloor = h$$



t is a full binary tree

$$N(t) = 2^{(h-1)+1} - 1 = 2^h - 1$$



$$2^h - 1 < n < 2^{h+1} - 1$$

+1 +1 +1

$$2^h < n+1 < 2^{h+1}$$

log2 log2 log2

$$\boxed{h < \log_2(n+1) < h+1}$$

$$\lfloor \log_2(n+1) \rfloor = h$$

QED

If a binary tree with height h has l leaves, then $l \leq 2^h$

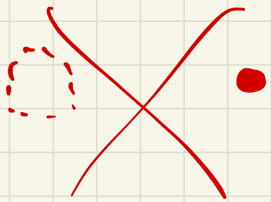
Strong Induction on the height of tree

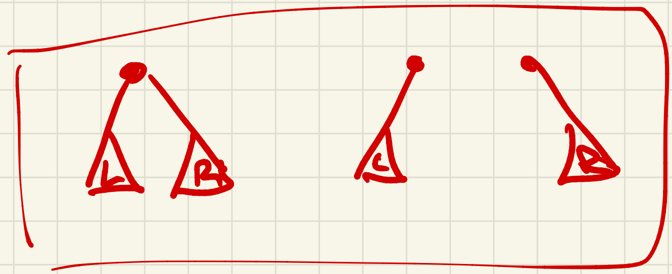
$P(0)$: $\circ$ $\bullet$ 1 leaf

$$1 \leq \underset{1}{\underline{2^0}} \quad \checkmark$$

$\forall k \leq h \; P(k) \implies P(h+1)$

for all trees T with height $k \leq h$ the number of leaves $l \leq 2^k \leq 2^h$ $\implies$ A tree T' with height $h+1$ the number of leaves $l' \leq 2^{h+1}$

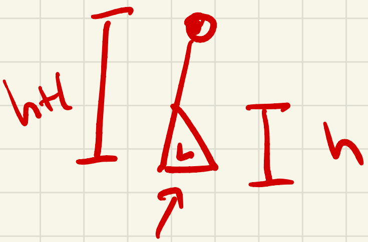




Cases a & b

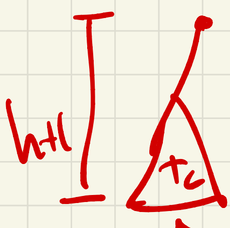
binary trees of different heights

Case:



T_L with height $h_L = h$

By IH, $\ell_L \le 2^h$

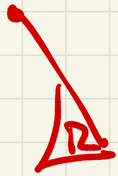


$$\boxed{\ell' \le 2^h \le 2^{h+1}}$$

All the leaves here!

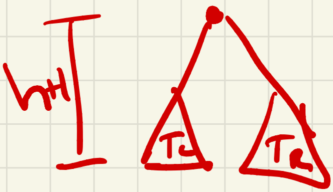
$\ell' = \ell_L$

Case:



— but treat tree same as other case where we swap L for R ✓

Case



T_L has height h_L

T_R has height h_R

$$\max(h_L, h_R)\cancel{+1} = h\cancel{+1}$$

$$h_L \leq h \qquad h_R \leq h$$

T_L and T_R are subject to IH

$$l_L \leq 2^h \qquad l_R \leq 2^h$$

$$l' = l_L + l_R \leq 2^h + 2^h \leq 2^{h+1}$$ ✓