CSCI 1311: Problem Set 6

Due: Apr. 21, 2020

Instructions:

Your submission must be **typed** and submitted to gradescope as a single pdf. You **may not** submit handwritten pictures via gradescope.

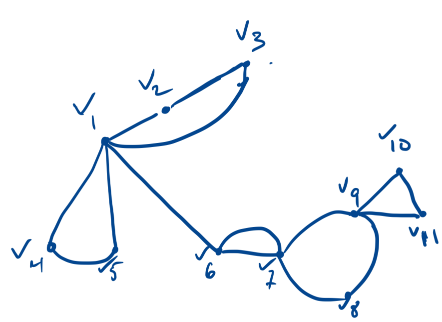
- You must include a cover page, that contains your name, the assignment information, the date, and your GW email address. No answers to questions should appear on the cover page.
- Try and organize your submission such that answers to questions (or parts of questions) do not span multiple pages. This will make it much easier to grade. **Ideally, each page will start with a new question (or part of question). See the sample for PS0 for a nice easy formatting.**
- On gradescope, be sure to mark which page your answer to each question (or sub question) is located. Doing so inaccurately could lead to issues with grading.

Question Weighting

Question:	1	2	3	4	5	Total
Points:	35	25	10	10	30	110

This problem set is graded out of 100 points, scores above 100 points are allowed and considered bonus.

1. Consider the following underacted graph, G.



   (a) **[5 points]** What is the matrix representation of the graph?

   (b) **[5 points]** Using the matrix representation, how many closed walks exist in the graph of length 3 that start/end at any given vertex?

(c) **[2 points]** Does the graph contain a Euler circuit? If so, why must it and provide such a circuit, if not, why not? For a negative answer, you should provide a proof; more than, it doesn't exist or you can't find one.

(d) **[3 points]** Does the graph contain a Euler trail? If so, why must it and provide such a circuit, if not, why not? For a negative answer, you should provide a proof; more than, it doesn't exist or you can't find one.

(e) **[5 points]** Does the graph contain a Hamiltonian Circuit? If so, provide it, otherwise, if not, why not? For a negative answer, you should provide a proof using an example from the graph; more than, it doesn't exist or you can't find one.

(f) **[5 points]** An *articulation point* (or *single point of failure*) of a connected graph is a node whose removal disconnects the graph. List the articulation points of G, if any.

(g) **[5 points]** Perform *two* Depth-First-Search traversal of the graph starting with vertex v_1 and v_7. Break ties by always choosing the edge that connect to a vertex of lower number first.

(h) **[5 points]** Perform *two* Breadth-First-Search traversals of the graph starting with vertex v_1 and v_7. Break ties by always choosing the edge that connects to a vertex of lower number first.

2. Consider the following simple graph $G = \{E, V\}$:

$$\begin{aligned} V =& \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \\ E =& \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_7\}, \{v_4, v_5\} \\ & \{v_5, v_7\}, \{v_6, v_7\}, \{v_7, v_8\}\} \end{aligned}$$

(a) **[10 points]** The *distance* function $d(u, v)$ between two vertices u and v is the length of the shortest path, as counted by the number of edges crossed in that path. Compute the distance matrix A based on G such that $d_{ij} = d(i, j)$. *Hint: consider BFS as a way to compute distances*

(b) **[2 points]** The *diameter* of a graph is the largest distance in that graph. What is the diameter of G?

(c) **[3 points]** The *radius* of the graph from a vertex v is the distance from v to the farthest node. For each vertex v in G, compute the radius of G from v.

(d) **[5 points]** The *center* of a graph is a vertex v where the radius of G from v is the smallest. If there are ties, the graph has multiple centers. Find the center(s) of G.

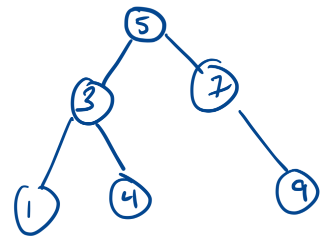
(e) **[5 points]** The *average radius* of the graph from a vertex u is the average distance from x to all the nodes of the graph.

For each vertex u in G, compute the average radius of G from u. Which vertex/vertices has/have the smallest average radius?

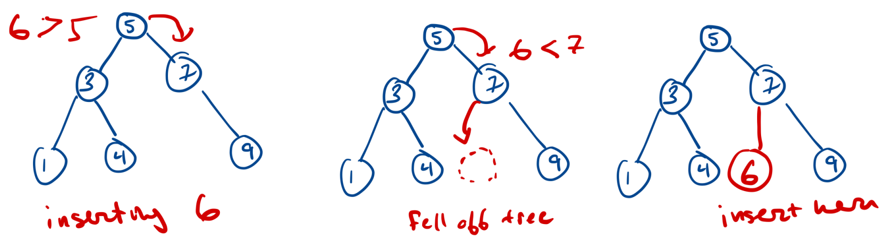
3. **[10 points]** Consider a graph where there exists a single center vertex v. The distance to that vertex is r. Prove that the diameter d of the graph is at most $d = 2r$. ***See the previous question for the definition of center and diameter of the***

4. **[10 points]** Let T_a and T_b be trees. Prove that if you create a single edge from any vertex in T_a to any vertex in T_b the resulting graph T_{ab} is also must be a tree.

5. A **binary search tree** (BST) is a rooted tree where (1) every vertex holds a numerical value (called a key), and (2) for any internal vertex in the tree (v), the keys of the vertices to the left of v is less-than-or-equal-to the key of v, the keys of the vertices to the right of v is greater-than-or-equal-to the key of v. For example, the following is a valid (non-full) BST:



If I were to insert a new vertex into the BST, say 6, I would start at root, and ask, is this vertex less than or equal to the given vertex? If it is greater than, I go right, if less than (or equal) I go left. Then I'd ask the same of the next vertex, and so on, until I fall off the tree; that's where the new vertex is inserted. Like in the visual below:



(a) **[10 points]** Prove (by induction on the number of vertices in the tree) that the largest key in a BST must always be associated with the right-most vertex. That is, the vertex that you reach by always taking the right branch from the root of the tree until you reach a leaf.

(b) **[20 points]** Prove (by induction on the height of the tree) that if the BST is a *perfect* or *full* binary tree with height $h \geq 0$, than the root of the tree must be median value of all the keys, that is it is middle value $n/2$ if the all the keys were aligned in a list of n values.